

Three classes of triangles

Solutions

1 Definitions and introductory problems

1. Applying the Ptolemy theorem to cyclic quadrilaterals OA_0CB_0 , OB_0AC_0 , OC_0BA_0 and summing the obtained equalities we have

$$Rp = d_1 \frac{b+c}{2} + d_2 \frac{c+a}{2} + d_3 \frac{a+b}{2},$$

where p is the semiperimeter of the triangle. Since $\frac{ad_1+bd_2+cd_3}{2} = S_{ABC} = pr$, this is equivalent to the desired equality.

Note. If we consider d_1, d_2, d_3 as the oriented distances this formula is also correct for an obtuse-angled triangle.

2. a) Note that $R^2 - OI^2 = CI \cdot C'I$ because this is the absolute value of the degree of I wrt Ω . Also we have that $CI = r/\sin \frac{C}{2}$, and $C'I = AI = BI = 2R \sin \frac{C}{2}$ by the trident theorem. From this we obtain the desired equality.

b) Considering the degree of I_c we obtain the similar proof.

3. Let the line passing through K and parallel to BC meet AB, AC at points X, Y respectively. Then γ is the excircle of triangle AXY homothetic to ABC . Points K and A_2 are correspondent in this homothety, thus these points and the homothety center A are collinear.

1A. Using the definition of A -triangle we obtain that the diameter perpendicular to OI meets the circumcircle at A' . By the trident theorem this point is the circumcircle of triangle IBC . Therefore constructing this circle we find two vertices of the desired triangle. The third vertex is the second common point of $A'I$ and the circumcircle.

2A. We can suppose that $AB \leq AC$. Let X be the projection of A' to AB , and L_a be the base of the bisector from A . Note that $\angle OA'I = \angle BA'X$ because $\angle A'BX = \angle A'CA = \sphericalangle ABA'/2 = \sphericalangle AB + \sphericalangle A'C/2 = \angle A'L_aC = \angle A'IO$. Since $A'B = A'I$ this yields that the distances from A' to AB, AC and OI are equal.

3A. We obtain this from the previous problem considering the homothety mapping Δ to ABC .

4A. By problem 1 the sum of distances from O to AB and AC is equal to R because $OA_0 = r$, this yields the desired assertion because $OC_0 = R \cos \angle C$, $OB_0 = R \cos \angle B$.

- 5A. Similarly to problem 1 we obtain that in an arbitrary triangle $r_a - R = d_2 + d_3 - d_1$. Summing this with the equality of problem 1 we obtain that $r + r_a = BH + CH$. In A -triangle this is equivalent to the desired equality by the previous problem.
- 6A. This assertion is true for an arbitrary triangle and follows from the similarity of triangles $A_0A'I$ and ZAI . In fact since $A_0A' = R(1 - \cos A) = 2R \sin^2 \frac{A}{2}$ we obtain that $AZ = A_0A' \cdot AI/A'I = r$.
- 7A. By the previous problem $AO \parallel A_0I$. Thus triangle $A'A_0I$ is similar to isosceles triangle $A'OA$, i.e. $OA_1 = IA_0 = A_0A' = R - r$. This also follows from problem 2: in fact $OA_1^2 = IA_1^2 + IO^2 = r^2 - 2Rr + R^2 = (R - r)^2$. The first equality is true because $OI \parallel BC$.
- 8A. Since $AO \parallel A_0I$, lines OH'_a and A_0I form equal angles with altitudes AH . But lines A_0I and OA_1 also form equal angles with AH because OA_0A_1I is a rectangle.
- 9A. Since $\angle IAH = |\frac{B-C}{2}|$ the desired assertion is equivalent to $2R \cos A \cos \frac{B-C}{2} = r / \sin \frac{A}{2}$ and to $\cos B + \cos C = 1$.
From problem 6A we have $AZ = r$, also $AH = 2OA_0 = 2r$ and $\angle IAZ = \angle IAO = \angle AOI$ because $AO \parallel IA_0$. From this $ZA = ZO = ZI$, which evidently yields the assertion of the problem.
- 10A. From problem 6A we obtain that line A_0I bisects segment AH , i.e. their common point Z lies on the Euler circle. Also since $OA \parallel A_0I$ we obtain that $\angle AIZ = \angle OAI = \angle IAZ$ and $IZ = AZ = r$. Therefore Z lies also on the incircle, i.e. Z coincide with F .
- 11A. For an arbitrary triangle the considered homothety center is isogonally conjugated to the Nagel point. Hence the desired assertion immediately follows from problem 3A.
- 12A. This evidently follows from problem 2A because $A'C'$ is perpendicular to the bisector of angle B .
- 1B. Since the circles are orthogonal $R^2 + 2Rr_b = OI_b^2 = R^2 + r_b^2$.
- 2B. a) Since $BA_b = p$, we have $R_b = p \operatorname{tg} \frac{B}{2} = R(\sin A + \sin B + \sin C) \operatorname{tg} \frac{B}{2}$. But $\sin A + \sin B + \sin C = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} + 2 \sin \frac{C}{2} \cos \frac{C}{2}$. Since $\frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}$, this is equal to $2 \cos \frac{C}{2} (\cos \frac{A-B}{2} + \cos \frac{A+B}{2}) = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$ which yields the desired equality.
- b) By the previous problem $1 = 2 \sin \frac{B}{2} \cos \frac{A}{2} \cos \frac{C}{2} = \sin \frac{B}{2} (\cos \frac{A-C}{2} + \sin \frac{B}{2}) = \frac{1}{2} (\sin \frac{B+A-C}{2} + \sin \frac{B+C-A}{2} + 1 - \cos B$. Since $\frac{B+A-C}{2} = \frac{\pi}{2} - C$ this is equivalent to the desired equality.

- 3B. In next problem we prove that line H_aH_c touches γ , i.e. γ is the excircle of triangle BH_aH_c , which is similar to the given triangle with coefficient $\cos B$. Hence $\cos B = r/r_b = (p-b)/p$ and $A_1C_b \perp BC$, $C_1A_b \perp AB$. Then by the Thales theorem lines AH_a and CH_c meet A_1C_1 at the same point.
- 4B. Since $H_aH_c = AC \cos B$ quadrilateral AH_cH_aC is circumscribed if and only if $AC + AC \cos B = AC \cos A + AC \cos B$, which coincide with the equality from problem 2B.
- 5B. Since $r_b = 2R$ and $r = r_b \cos B$ we obtain that $OB_0 = r/2$, which evidently yields the desired assertion. Note also that the reflection of I about O lies on the perpendiculars from A_a , B_b , C_c to the correspondent sidelines. Therefore in B -triangle this point coincide with B_b .
- 1C. Let $R = r_c = 1$. Then by problem 2 $OI_c = \sqrt{3}$. The centers of the circles and their common points form a rhombus, therefore its second diagonal is equal to 1.
- 2C. **Answer.** $\pi/4$, $\pi/4$, $\pi/2$.
Hint. Use the next problem.
- 3C. The solution is similar to problem 2B.
- 4C.
- 5C. Triangles II_aI_b and $C_cB_cA_c$ are homothetic because their sidelines are parallel to two internal and one external bisector of the given triangle. Also triangle ABC is the orthotriangle of triangle II_aI_b . Thus the circumradius of triangle II_aI_b is equal to $2R$, and its circumcenter is the reflection of I_c in O . Hence the homothety center lies on segment I_cO and divide it in ratio 2 : 1.
- 6C. It is easy to see that $\angle OAI_c = \angle AI_cA_c$. Also $OA = I_cA_c$. Thus OAI_cA_c is an isosceles trapezoid and $AA_c = OI_c = R\sqrt{3}$. Similarly two remaining segments have the same length.
- 7C. The angle between AA_c and BB_c is equal to $2\angle AI_cB$ because these lines are symmetric to OI_c wrt the perpendicular bisectors to segments AI_c and BI_c respectively.
- 8C. Immediately follows from problem 5C. Also this can be obtained from problem 6C considering the correspondent isosceles trapezoids.